Note 1.
\[ B = \text{set of Bounded functions} \]
\[ C = \text{set of Continuous functions} \]
\[ D = \text{set of Differentiable functions} \]
\[ D^n = \text{set of } n \text{ times Differentiable functions} \]
\[ C^n = \text{set of } n \text{ times Continuously Differentiable functions} \]
\[ R = \text{set of Riemann Integrable functions} \]
\[ P = \text{set of all possible Partitions of an interval} \]
\[ LC = \text{set of Lipchitz Continuous functions} \]
\[ UC = \text{set of Uniformly Continuous functions} \]

B. Real Analysis

C. Numerical Analysis

Further Real Analysis required

1 Preliminaries

Axiom 1. Completeness Axiom
If \( A \) is a bounded non-empty subset of real numbers \( \mathbb{R} \) then \( \text{sup} \ A \) and \( \text{inf} \ A \) exists.

Theorem 1. Intermediate Value Theorem
Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Let \( c \) be between \( f(a) \) and \( f(b) \). Then there exists \( x \in (a, b) \) such that \( f(x) = c \).

Theorem 2. Extreme Value Theorem
Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then \( f \) attains its maximum and minimum on \( [a, b] \) i.e. there exists \( c, d \in [a, b] \) such that \( f(c) = \max \{ f(x) | x \in [a, b] \} \) and \( f(d) = \min \{ f(x) | x \in [a, b] \} \).

Theorem 3. Mean Value Theorem
Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \). Then there exists \( \zeta \in (a, b) \) such that \( \frac{f(b) - f(a)}{b-a} = f'(\zeta) \)

Theorem 4. Generalized Mean Value Theorem
Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \) and \( g' \neq 0 \) on \( (a, b) \). Then there exists \( \zeta \in (a, b) \) such that \( \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\zeta)}{g'(\zeta)} \)

Theorem 5. L'Hopital Rule
Let \( f, g \in D, f(a) = g(a) = 0 \) and \( g' \neq 0 \) except possibly at \( a \).
If \( \lim_{x \to a} \frac{f(x)}{g(x)} = L \) then \( \frac{f'(x)}{g'(x)} = L \).

Note 2.
1. It is convenient to assume \( f \in D \) in a larger open interval containing \( (a, b) \) for Mean Value Theorems.
2. L'Hopital rule applies in all other cases of limits of \( x \) and even when \( f(x), g(x) \rightarrow 0 \) or \( \infty \) at the limit with conditions suitably modified.
2 Riemann Integral

**Definition 1. Riemann Integral**

Let \( f \) be bounded on \([a, b]\). i.e. \( f \in \mathcal{B}[a, b] \)

Let \( P = \{x_0, x_1, \cdots, x_n\} \) with \( x_0 = a, x_n = b \) and \( \Delta x_k = x_k - x_{k-1} > 0 \) for \( 1 \leq k \leq n \)
be a partition of \([a, b]\), i.e. \( P \in \mathcal{P}[a, b] \).

Let \( M_k = M_k(P, f) = \sup \{ f(x) | x \in [x_{k-1}, x_k] \} \) and

\( m_k = m_k(P, f) = \inf \{ f(x) | x \in [x_{k-1}, x_k] \} \)

Define Upper Riemann sum \( U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k \) and

Lower Riemann Sum \( L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k \)

Define Upper Riemann Integral \( U(f) = \inf \{ U(P, f) | P \in \mathcal{P}[a, b] \} \) and

Lower Riemann Integral \( L(f) = \sup \{ L(P, f) | P \in \mathcal{P}[a, b] \} \).

Iff \( U(f) = L(f) \) we say that \( f \) is Riemann Integrable on \([a, b]\). i.e. \( f \in \mathcal{R}[a, b] \) and write \( \int_{a}^{b} f(x)dx \) for the common value and call it the Riemann integral of \( f \) on \([a, b]\).

**Note 3.** It is clear that \( m_k \leq M_k \) therefore \( L(P, f) \leq U(P, f) \)

We will later show that even \( L(f) \leq U(f) \) is true.

**Example 1.**

1. Consider the function \( f(x) = 1 \) if \( x \in \mathbb{Q} \) and \( 2 \) if \( x \in \mathbb{R} - \mathbb{Q} \). Is \( f \in \mathcal{R}[a, b] \)?

2. Consider equispaced partitions of \([a, b]\) for the function \( f(x) = e^x \). Show that \( U(f) = L(f) = e^b - e^a \) without direct integration.

**Theorem 6.** Let \( P, P^* \in \mathcal{P}[a, b] \).

We say that \( P^* \) is a refinement of \( P \) iff \( P \subset P^* \)

We have \( L(P^*, f) \geq L(P, f) \) and \( U(P^*, f) \leq U(P, f) \)

Also \( L(f) \leq U(f) \) for any \( P \in \mathcal{P}[a, b] \)

**Theorem 7.** Riemann condition for Riemann Integrability

\( f \in \mathcal{R}[a, b] \) iff \( \forall \epsilon > 0, \exists P \in \mathcal{P}[a, b]; U(P, f) - L(P, f) < \epsilon \)

**Theorem 8.** If \( f, g \in \mathcal{R}[a, b] \) then

1. \( f + g \in \mathcal{R}[a, b] \) and \( \int_{a}^{b} (f(x) + g(x))dx = \int_{a}^{b} f(x)dx + \int_{a}^{b} g(x)dx \)

2. \( fg \in \mathcal{R}[a, b] \)

3. If \( f \leq g \) on \([a, b]\) then \( \int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx \)

4. \( |f| \in \mathcal{R}[a, b] \) and \( \int_{a}^{b} |f(x)dx| \leq f_{a}^{b} |f(x)|dx \)

5. If \( c \in (a, b) \) then \( f \in \mathcal{R}[a, c], f \in \mathcal{R}[c, b] \) and \( \int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx \)

**Theorem 9.** Let \( f \in \mathcal{R}[a, b], P \in \mathcal{P}[a, b] \) and \( t_k \in [x_{k-1}, x_k] \)

Then \( |\sum_{k=1}^{n} f(t_k)\Delta x_k - \int_{a}^{b} f(x)dx| \leq U(P, f) - L(P, f) \)

**Theorem 10.** Fundamental Theorem of Calculus

If \( f \in \mathcal{R}[a, b] \) and there exists \( F \in \mathcal{D}[a, b] \) such that \( f = F' \)
then \( \int_{a}^{b} f(x)dx = F(b) - F(a) \).

**Definition 2. Strong forms of Continuity**

1. \( f \) is uniformly continuous on \( A \) i.e. \( f \in \mathcal{UC}(A) \)

iff \( \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in A; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \)
2. \( f \) is Lipchitz continuous on \( A \) i.e. \( f \in \mathcal{LC}(A) \) iff \( \exists L > 0, \forall x, y \in A; |f(x) - f(y)| < L|x - y| \nabla \)

**Theorem 11.** \( f \in \mathcal{LC}(A) \Rightarrow f \in \mathcal{UC}(A) \Rightarrow f \in \mathcal{C}(A) \)

**Theorem 12.** \( f \in \mathcal{C}[a, b] \Rightarrow f \in \mathcal{UC}[a, b] \Rightarrow f \in \mathcal{R}[a, b] \)

**Example 2.** Show that \( \frac{1}{x} \) is not uniformly continuous on \( (0, 1] \) but \( x^2 \) is.

**Theorem 13.** Second Fundamental Theorem of Calculus
Let \( f \in \mathcal{R}[a, b], x \in [a, b] \) and \( F(x) = \int_a^x f(t)dt \). If \( f \in \mathcal{R}[a, b] \) then \( F \in \mathcal{C}[a, b] \). If \( s \in (a, b) \) and \( f \in \mathcal{C}(s) \) then \( F \in \mathcal{D}(s) \) and \( F'(s) = f(s) \)

**Theorem 14.** Integration by Parts
If \( F, G \) differentiable on \( [a, b], F' = f \in \mathcal{R}[a, b] \) and \( G' = g \in \mathcal{R}[a, b] \). Then \( \int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx \)

**Theorem 15.** Change of Variable
\( g \) has continuous derivative \( g' \) on \( [c, d] \). \( f \) is continuous on \( g([c, d]) \) and let \( F(x) = \int_{g(c)}^x f(t)dt, x \in g([c, d]) \). Then for each \( x \in [c, d], \int_c^x f(g(t))g'(t)dt \) exists and has value \( F(g(x)) \).

**Theorem 16.** Mean Value Theorem for Integrals
\( f \in \mathcal{C}[a, b] \). Then there exists \( \zeta \in (a, b) \) such that \( \int_a^b f(x)dx = f(\zeta)(b - a) \).

**Theorem 17.** Generalized Mean Value Theorem for Integrals
\( f \in \mathcal{C}[a, b], g \in \mathcal{R}[a, b] \) and \( g \) does not change sign on \( [a, b] \). Then there exists \( \zeta \in (a, b) \) such that \( \int_a^b f(x)g(x)dx = f(\zeta)\int_a^b g(x)dx \).

3. Improper Riemann Integral

**Definition 3.** Improper Integrals of the First Kind
Suppose \( \int_a^b f(x)dx \) exists for each \( b \geq a \) Iff \( \lim_{b \to \infty} \int_a^b f(x)dx \) exists and equal to \( I \notin \mathbb{R} \) we say that \( \int_a^\infty f(x)dx \) converges to the value \( I \) and diverges otherwise.

**Theorem 18.** Direct Comparison Test for Integrals
Let \( f, g \in \mathcal{R}[a, b] \) for all \( b > a \) and \( f(x) \leq g(x) \) for all \( x > a \). If \( \int_a^\infty g(x)dx \) converges, then \( \int_a^\infty f(x)dx \) converges.

**Theorem 19.** Limit Comparison Test for Integrals
Let \( f, g \in \mathcal{R}[a, b] \) for all \( b > a \) and \( f(x), g(x) > 0 \) for all \( x > a \). If \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \), then if
1. \( c \neq 0, \infty \) then \( \int_a^\infty f(x)dx \) conv. \( \iff \int_a^\infty g(x)dx \) conv.
2. \( c = 0 \) and \( \int_a^\infty g(x)dx \) conv. \( \iff \int_a^\infty f(x)dx \) conv.
3. \( c = \infty \) and \( \int_a^\infty g(x)dx \) div. \( \iff \int_a^\infty f(x)dx \) div.
Definition 4. Other types of Improper Integrals

\[ f : [a, \infty) \to \mathbb{R} : \int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx \]

\[ f : (-\infty, b] \to \mathbb{R} : \int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx \]

\[ f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx, c \in \mathbb{R} \]

\[ f : (a, b] \to \mathbb{R} : \int_{a}^{b} f(x)dx = \lim_{t \to a^+} \int_{t}^{b} f(x)dx \]

\[ f : [a, b) \cup (c, b] \to \mathbb{R} : \int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx \]

Note 4. Similar types of convergence tests exists for above types of improper integrals.

Example 3.
1. Find \( \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx \) if it exists.
2. Prove that if \( f \) is bounded above and increasing then \( \lim_{x \to \infty} f(x) \) exists and finite.
3. Prove that \( \int_{a}^{\infty} |f(x)| dx \text{ conv.} \Rightarrow \int_{a}^{\infty} f(x)dx \text{ conv.} \)
4. Prove that if \( |f(x)| \leq Me^{ax} \), then the Laplace Transform of \( f(x) \), \( F(s) = \int_{0}^{\infty} f(x)e^{-sx}dx \) exists for all \( s > a \).

Example 4. Gamma function is defined by \( \Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt \). Show that
1. \( F(x) \) exists iff \( x > 0 \).
2. \( \Gamma(x) = (x-1)\Gamma(x-1), x > 1 \).
3. \( \Gamma(n) = (n-1)! \) for all integer \( n \geq 1 \).
4. we can use (2) to define \( \Gamma(x) \) for \( x < 0 \).
5. \( \Gamma(x) \) does not exist for \( x = 0, -1, -2, -3, \ldots \)
6. Show that \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and \( \int_{0}^{\infty} e^{-t^2}dt = \sqrt{\pi} \) using \( \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \)
7. Use the formula for the \( n \) dimensional ball \( V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)} \) to find the volumes of
   2, 3, 4, 5 dimensional balls.
8. Show that \( \Gamma(x) \) is continuous on \((0, \infty)\)
9. Prove that the Beta Function \( B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1}dt \) exists for all \( x, y > 0 \).
   It can be shown that \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \).

Example 5. 1. Find the range of convergence of the following special functions called Fresnel Integrals \( S(x) = \int_{0}^{x} \sin(t^2)dt \) and \( C(x) = \int_{0}^{x} \cos(t^2)dt \).
2. Find the range of convergence of the Logarithmic Integral \( li(x) = \int_{0}^{x} \frac{dt}{\log t} \).

Example 6. The exponential integral is defined by \( Ei(x) = -\int_{-x}^{\infty} e^{-t}dt \). We are interested in the related integral \( F(x) = \int_{1}^{x} \frac{e^{-t}}{t}dt \) which is equal to \( Ei(-x) - Ei(-1) \).
1. Show that \( F(\infty) \) is a converging improper Riemann integral.
2. Show that \( F(0) \) is a diverging improper Riemann integral.
3. Write the 2nd degree \((n = 2)\) Taylor polynomial and the integral form of the remainder of the Taylor series of \( F(x) \) at \( a = 1 \).
4. What can be the radius of convergence of the Taylor series of \( F(x) \) at \( a = 1 \)?
Direct proof is not needed, use the previous.
4 Taylor Series with Remainder

**Theorem 20.** Taylor series of \( f \in D^{n+1} \) at \( a \).

\[
f(x) = T_n(x, a) + R_n(x, a)
\]

Taylor Polynomial \( T_n(x, a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k \)

Lagrange Remainder \( R_n(x, a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - a)^{n+1} \)

where \( \zeta \) between \( x \) and \( a \)

**Proof.** Use Generalized Mean Value Theorem on \( F(t) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(x - t)^k \) and \( G(t) = (x - t)^{n+1} \).

**Example 7.** Let \( f(x) = \ln(1 + x) \). Show that

1. \( T_n(x, 0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k \)
2. Find the range of convergence of \( x \)
3. Show that \( R_n(x, 0) \to 0 \) as \( n \to \infty \) for \( x \geq 0 \)
4. Find the value of \( \ln(1.5) \) accurate to 0.000001

**Theorem 21.** Integral form of the Remainder. \( f \in C^{n+1} \)

\[
R_n(x, a) = \frac{1}{n!} \int_{x}^{a} f^{(n+1)}(t)(x - t)^n dt
\]

**Example 8.**

1. Show that \( R_n(x, 0) \to 0 \) as \( n \to \infty \) for \( x < 0 \)
2. Find the value of \( \ln(0.2) \) accurate to 0.000001

**Theorem 22.** Some Theorems on Riemann Integrals

If \( f \in C[a, b] \) then \( f \in R[a, b] \)

If \( f \in R[a, b] \) and \( F(x) = \int_{a}^{x} f(t)dt \) then \( F \in C[a, b] \)

If \( f \in C[a, b] \) and \( F(x) = f_{a}^{x} f(t)dt \) then \( F \in D[a, b] \) and \( F' = f \)

**Theorem 23.** Generalized Mean Value Theorem for Riemann Integrals

Let \( f \in C[a, b] \) and \( g \in R[a, b] \) and \( g \) does not change sign on \([a, b] \).

Then there exists \( \zeta \in (a, b) \) such that \( \int_{a}^{b} f(t)g(t)dt = f(\zeta) \int_{a}^{b} g(t)dt \)

**Theorem 24.** Other forms of Remainders. \( f \in C^{n+1} \)

\[
R_n(x, a) = \frac{f^{(n+1)}(\zeta)}{n!(p+1)} (x - \zeta)^{n-p}(x - a)^{p+1}, \quad 0 \leq p \leq n
\]

\( p = n \), Lagrange Remainder \( R_n(x, a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - a)^{n+1} \)

\( p = 0 \), Cauchy Remainder \( R_n(x, a) = \frac{f^{(n+1)}(\zeta)}{n!} (x - \zeta)^n(x - a) \)

where \( \zeta \) between \( x \) and \( a \)

**Example 9.**

1. Find the value of \( \ln(0.02) \) accurate to \( 10^{-6} \) using the Taylor series expansion at 0 for \( x < 0 \).
2. Find the values of \( \ln 2, \ln 5, \ln 500, \ln(0.2), \ln(0.02) \) accurate to \( 10^{-6} \) using the Taylor series expansion at 0 for \( x > 0 \).
3. Find the Taylor series expansions for \( e^x, \sin x, \tan^{-1}x, \tan x \) at 0 with remainder.
4. Find the range of convergence.
5. Show that the remainder \( R_n(x, a) \to 0 \) as \( n \to \infty \) within the range of convergence.
6. Find the values of $e$, $\sin 1$, $\tan^{-1}1$ accurate to the 10th decimal place.
7. Find the values of $e^4$, $\sin 4$, $\tan^{-1}4$ accurate to the 6th decimal place using a suitable Taylor series expansion.
8. Deduce the values of 7. from 6. whenever it is possible.
9. Show that $e$ is irrational.

Example 10. Consider the function $f(x) = x^{1/10}$
1. Write down the nth degree Taylor Polynomial near $c > 0$.
2. Show that the remainder satisfies $|R_n(x, c)| < \begin{cases} \frac{x^{1/10}}{10(n+1)} (\frac{x-c}{c})^{n+1} & x > c > 0 \\ \frac{c^{1/10}}{10(n+1)} (\frac{c-x}{c})^{n+1} & c > x > 0 \end{cases}$
3. Show that the value of $1000^{1/10}$ accurate to 3 decimal places is 1.995.
4. Find the value of $1025^{1/10}$ accurate to 10 decimal places.

Definition 5. Power Series
An infinite series of the form $\sum_{k=0}^{\infty} u_k(x)$ is
1. Converges point-wise to $S(x)$ iff $\forall \varepsilon > 0 \forall x \exists N > 0 \forall n < N \Rightarrow |\sum_{k=n}^{\infty} u_k(x) - S(x)| < \varepsilon$
2. Converges Uniformly to $S(x)$ iff $\forall \varepsilon > 0 \exists N > 0 \forall x \forall n < N \Rightarrow |\sum_{k=n}^{\infty} u_k(x) - S(x)| < \varepsilon$

Definition 6. A Power Series at $a$ is an infinite series of the form $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$.

Theorem 25.
1. There exists $R \in [0, \infty]$ called the radius of convergence such that a power series converges absolutely and uniformly for $|x-a| < R$ and diverges for $|x-a| > R$.
2. The power series may converge conditionally or diverge for $|x-a| = R$.
3. Radius of convergence $R = \lim_{k \to \infty} |a_k|^{1/k}$
4. Since the power series is uniformly converges for $x \in (a-R, a+R)$ the series may be differentiated term-by-term giving $S'(x) = \sum_{k=1}^{\infty} k a_k(x-a)^{k-1}$ with the same radius of convergence.
5. Since the power series is uniformly converges for $x \in (a-R, a+R)$ the series may be integrated term-by-term giving $\int_{a}^{x} S(t)dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-a)^{k+1}$ with the same radius of convergence.

Example 11. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
1. Write down the length of the perimeter $C(a, b)$ as a definite integral.
2. Convert the above integral into the Incomplete Elliptic Integral of the Second Kind $E(\phi, k) = \int_{0}^{\phi} \sqrt{1 - k^2 \sin^2 x} dx$.
3. What is $C(a, a)$?
4. Use Taylor Series to calculate $C'(a, 2a)/C(a, a)$ accurate to 0.001.

Example 12. Consider a pendulum of mass $m$ and length $\ell$ oscillating at an angle $2\alpha$ in a gravitational field of strength $g$.
1. Write the time period $T(\alpha)$ as a definite integral.
2. Convert the above integral into the Incomplete Elliptic Integral of the First Kind...
Theorem 26. Second Derivative Test

\[ f \in C^1. \]  
1. If \( f''(a) > 0 \) then \( a \) is a local minimum of \( f \).
2. If \( f''(a) < 0 \) then \( a \) is a local maximum of \( f \).

**Proof.** Use the second order Taylor series \( f(a + h) = f(a) + f'(a)h + \frac{1}{2}f''(a + \theta h) \).

Note that the first two terms are the Tangent Line at \( a \). Since \( f'(a) = 0 \) we have \( f(a + h) - f(a) = \frac{1}{2}f''(a + \theta h) \). Since \( f'' \in C \) and \( f''(a) > 0 \), we can select \( h \) such that \( f''(a + \theta h) > 0 \) for all \( \theta \in (0, 1) \). \( \square \)

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**Formula:** \((ALPHAX + 1)2x^2ALPHAX - 10x^6CALCX?10 =\)

**Sum:** \( \text{SHIFT} \sum \Box \Box \Box : \sum((-1)x^2(ALPHAX - 1)(1/2)x^2ALPHAX \div ALPHAX, 1, 10) \)

**Mathematica 1.**

**Formula:** \( f[n_] := (1 + n)2^n - 10^6; \text{Table}[[\{n, f[n]\}], \{n, 1, 50\}] \)

**Sum:** \( \text{Sum}((-1)\Lambda(k - 1)(1/2)\Lambda k/k, \{k, 1, 10\}] \)

**Taylor Series:** \( \text{Series}[\text{Log}[1 + x], \{x, 0, 20\}] \)

5 Numerical Integration

Theorem 27. Trapezoidal Rule

\[ f \in C^2[a, b], h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \leq k \leq n, \zeta \in (a, b) \]

\[ \int_a^b f(x)dx = \frac{h}{2} \left[ f(x_0) + 2 \sum_{k=1}^{n-1} f(x_k) + f(x_n) \right] - \frac{(b-a)^3}{12n^2} f''(\zeta) \]

Theorem 28. Simpson’s Rule (see Proofs.pdf for the proof for the error)

\[ f \in C^4[a, b], n \text{ is even}, h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \leq k \leq n, \zeta \in (a, b) \]

\[ \int_a^b f(x)dx = \frac{h}{3} \left[ f(x_0) + 4 \sum_{k=1}^{n/2} f(x_k) + 2 \sum_{k=0}^{n/2} f(x_k) + f(x_n) \right] - \frac{(b-a)^5}{180n^4} f^{(4)}(\zeta) \]

Example 13.

1. For each of the following integrals, use the Trapezoidal and Simpson’s rules to find the number of divisions needed to find its value accurate to 0.001 and find the integral to that accuracy.

\[ \int_0^1 \sin(x^2)dx \]
\[ \int_0^1 \cos(x^2)dx \]
\[ \int_0^1 e^{-x^2}dx \]
\[ \int_0^\pi \sqrt{2 - \cos^2x}dx \]
\[ \int_{2}^{10} \frac{x}{\log x} \, dx \]

2. Derive a numerical integration rule and its error that uses the function value at the mid point (Mid Point Rule), left end point, right end point of each interval.

3. Use Mid Point Rule rule to do the integrals in Q1.

4. Show directly that cubic polynomials are integrated exactly (error is 0) by the Simpson’s rule.

5. Use Taylor series to derive an approximate formula for the remainder in Trapezoidal rule.

6. Use integration by parts to prove the error formula for the Simpson’s rule.

Example 14.

One method of doing numerical integration is Gaussian Quadrature. Note that both the Trapezoidal and the Simpson’s rules looks like

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{k} w_{k} f(x_{k}) \]

and we knew \( x_{k} \) and found \( w_{k} \). In this method we find both \( x_{k} \) and \( w_{k} \) so that the integral and the sum are equal for a given \( n \) degree polynomial \( p(x) \). It is achieved by forcing both sides equal for each power of \( x^{j} \) for \( j = 0, 1, 2, \ldots, n \). What is the degree of the polynomial we need to use if we want 3 points and the corresponding 3 weights? Find them for \([a, b] = [-1, 1]\) and use it to approximate integrals given above.

Casio fx-991ES uses a variant of this method.

Example 15. We want to approximate the value of

\[ F(2) = \int_{1}^{2} e^{-t} \, dt \]

accurate to 0.001.

1. Consider the method of term-by-term integration after using the Taylor series of \( e^{-t} \) at \( a = 0 \). What is the degree of the Taylor polynomial \( (n) \) that we have to use? Note that \( f^{(n+1)}(\zeta) \) in the error term is a continuous function of \( t \).

2. Find the value of \( F(2) \) accurate to 0.001 by this Taylor series method. Use the sum function in the calculator and confirm your answer by integrating function in the calculator.

3. Consider method of the Trapezoidal rule. How many intervals \( (n) \) needed?

4. Find the value of \( F(2) \) accurate to 0.001 by this Trapezoidal method. Use the sum function in the calculator and confirm your answer by integrating function in the calculator.

Casio 2. \( \int (f(X), a, b, m) \) and the default variable is \( X \) and \( n = 2^{m} \) for the Simpson’s method in model fx-991MS

Mathematica 2. \( \text{NIntegrate}[f(x), \{x, a, b\}, \text{Method} \rightarrow \text{TrapezoidalRule}] \)

6 Interpolation

Theorem 29. Lagrange Method of finding the Interpolating Polynomial \( p(x) \) of \( f(x) \) for the points \( x_{k}, 0 \leq k \leq n \)

\[ w_{j}(x) = \prod_{i=0, i \neq j}^{n} (x - x_{i}) \]

\[ w(x) = \prod_{i=0}^{n} (x - x_{i}) \]

\[ \ell_{j}(x) = \frac{w_{j}(x)}{w(x)} = \prod_{i=0}^{n} \frac{(x - x_{i})}{(x - x_{j})(x - x_{i})} = \frac{w(x)}{(x - x_{j})w'(x_{j})} \]

and \( \ell_{j}(x_{k}) = 0 \) if \( j \neq k \) and \( 1 \) if \( j = k \).
Example 16. Consider the data set \( A = \{ (2, 1), (3, 2), (4, 3), (6, 4) \} \).
1. Show that that the data set may be generated by the function \( f(x) = 4 \sin^2(\frac{x}{12}) \). Find an upper bound for the error.
2. For the same function on \([0, 6]\), find the number of points required to make the error \( \leq 0.001 \) and find the Interpolating Polynomial.
3. Use the error formula for the interpolating polynomial to derive the error formula for the Trapezoidal Method. Can you do the same with the Simpson’s Method?

Example 17. Consider the data set \( A = \{ (2, 1), (3, 2), (4, 3), (6, 4) \} \).
1. Find the Interpolating polynomial by direct matrix inversion.
2. One way of finding the Lagrange polynomial is to define it as the iterative process \( p(x) = p_0(x)(x - x_0) + p_0(x)(x - x_1) + q_1 \) and so on. See why this method is working and find the Interpolating Polynomial for \( A \). Mathematica seems to use this method.
3. Another method of finding the Interpolating Polynomial is to use the Newton’s divided differences. For \( x_0, x_1, x_2 \) we define \( f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) and \( f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \) and so on and the interpolating polynomial is given by \( p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \). Use the method in (2) to see why the formula is working and use it to find the interpolating polynomial for \( A \).
4. Find a polynomial (Hermite polynomial) that goes through the above points of \( A \) and satisfying \( p'(1) = 0, p'(2) = 1, p'(3) = 0 \).

Example 18.
1. Get the value of USD/LKR for the 1st of every month for this year 2018. (Write the values).
2. Use the Interpolating Polynomial to predict this value on 1st November 2018, 1st December 2018 and 1st January 2019. (Write the polynomial and the predicted values. You can use a software, write the code).
3. For the data set \( \{(x_k, y_k)\}, k = 0, \ldots, n \) a natural cubic spline is a twice differentiable piece-wise cubic polynomial \( p(x) \) which satisfies \( p(x_k) = y_k \) with \( p''(x_0) = p''(x_n) = 0 \). Let \( p(x) = \sum_{k=1}^{n} p_k(x) \) where \( p_k(x) \) is the part of \( p(x) \) on \([x_{k-1}, x_k]\) which is 0 elsewhere. Assume that \( p_k(x) = a_k(x - x_k)^3 + b_k(x - x_k)^2 + c_k(x - x_k) + d_k \) and that \( \Delta x_k = x_k - x_{k-1} = h \) is a constant.
   Derive the formula \( s_{k+1} + 4s_k + s_{k-1} = \frac{6}{h^2} (y_{k+1} - 2y_k + y_{k-1}) \); \( k = 1, \ldots, n-1 \) where \( s_k = p''(x_k) \).
   Also write the system of equations in matrix form that must be solved to find \( s_k \).
4. Use Cubic Spline to predict the same values (Write the calculated \( s_k \) values, the last cubic polynomial \( p_n(x) \) and the predicted values. You can use a software, write the code).

Mathematica 3. \texttt{InterpolatingPolynomial[\{(2, 1), (3, 2), (4, 3), (6, 4)\}, x]}
7. Numerical solutions of non-linear equations of one variable

**Algorithm 1. Bisection Method**
1. Find \( a_0, b_0 \) such that \( f(a_0)f(b_0) < 0 \).
2. \( k = 0 \).
3. \( x_k = \frac{a_k + b_k}{2} \).
4. If \( f(x_k) = 0 \) then stop and return \( x_k \).
5. If \( f(x_k)f(a_k) > 0 \) then \( a_{k+1} = x_k \) and \( b_{k+1} = b_k \) \( \text{else} \ b_{k+1} = x_k \) and \( a_{k+1} = b_k \).
6. If \( |b_k - a_k| < \epsilon \) then stop and return \( x_k \) \( \text{else} \ k \leftarrow k + 1 \) and goto 3.

**Theorem 30. Convergence of the Bisection Method**
1. \( f : [a, b] \to \mathbb{R} \)
2. \( f \) is continuous (i.e., \( f \in C \)).
3. \( a_0, b_0 \in [a, b] \) and we select \( a_k, b_k, x_k \) for \( k \geq 0 \) according to the above algorithm.
   Then
   \( \lim_{k \to \infty} x_k = \lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = z \in [a, b] \) is a root of \( f \).
2. \( |x_k - z| \leq \frac{1}{2}|b_k - a_k| = \left(\frac{1}{2}\right)^{k+1}|b_0 - a_0| \)

**Casio 3. Bisection Method**
\( \text{ALPHA X ALPHA} = \left( \text{ALPHA A + ALPHA B} \right) \div 2 \) \( \text{ALPHA : ALPHA X-} \)
- \( \text{ALPHA X CALC} \)

**Mathematica 4. Bisection Method**

**Algorithm**
\[ f[x_] := x - \text{E}(-x); a = 0; b = 1; \text{For}[k = 0, k < 20, k++, \{x = (a + b)/2, Print[N[k, a, b, x, f[x], Abs[a - b]/2, 10]], If[f[x] == 0, k = 20, If[f[x] > 0, b = x, a = x]]} \]  

**Builtin function**
\( \text{FindRoot}[f[x] == 0, \{x, 0\}] \), by iterations starting \( x_0 = 0 \)

**Theorem 31.** \( f : A \to \mathbb{R} \)
\( f \in C^1(A) \Rightarrow f \in \mathcal{L}(A) \) when \( A \) is closed with \( L = \max\{|f'(x)| : x \in A\} \) using Mean Value and Extreme Value theorems on \( f' \).

**Definition 7. Cauchy Sequence and Completeness.** Let \( u : \mathbb{N} \to A \) be a sequence.
1. \( u \) is a Cauchy sequence on \( A \) iff \( \forall \epsilon > 0, \exists N > 0, \forall n, m > 0 ; m, n > N \Rightarrow |u(m) - u(n)| < \epsilon \)
2. \( A \) is Complete iff Every Cauchy sequence on \( A \) is converging to a point of \( A \).

**Theorem 32.**
1. All converging sequences are Cauchy.
2. \( \mathbb{R}^n \) is complete.
3. A closed subset of a complete space is complete.
4. A complete space is closed.
Definition 8.
1. \( g \) is a Contraction iff it is Lipchitz continuous with Lipchitz constant \( L < 1 \).
2. \( z \) is a Fixed Point of \( g \) iff \( z = g(z) \).

Theorem 33. Global Convergence of the Fixed Point method (Banach Fixed Point Theorem)
1. \( g \colon [a, b] \rightarrow [a, b] \)
2. \( g \) is a contraction with Lipchitz constant \( L \)
3. \( x_0 \in [a, b] \) and \( x_{k+1} = g(x_k), k \geq 0 \)

Then
1. \( \lim_{k \to \infty} x_k = z \in [a, b] \) is a unique fixed point of \( g \)
2. \( |x_k - z| \leq \frac{L^k}{1-L}|x_1 - x_0| \)

Theorem 34. Local convergence of the Fixed Point method
Let \( z = g(z) \) be a fixed point. If \( g \in C^1 \) with \( |g'(z)| < 1 \), then there exists a neighbourhood of \( z \) such that the fixed point method is converging.

Algorithm 2. Fixed Point Method
1. Select \( x_0 \)
2. \( k = 0 \).
3. \( x_{k+1} = g(x_k) \)
4. If \( |x_{k+1} - x_k| < \epsilon \) then stop and return \( x_k \) else goto 3

Example 19. Consider the equations \( x = e^{-x} \) and \( x^5 - x - 1 = 0 \). For each case
1. Find intervals that contains real roots.
2. Find number of iterations needed to find each root to an accuracy of 0.0001 using each of the methods Bisection/Fixed Point
3. Do the iterations and find all real roots.

Example 20. Let \( T_n(x) = \sum_{k=1}^{n} \frac{(-x)^k}{k!} \) be the \( n \)th degree Taylor polynomial of \( e^{-x} \) at \( x = 0 \) and \( \lim_{x \to \infty} T_n(x) = e^{-x} \). Solve \( x = T_2(x) \) and find an approximate solution to \( x = e^{-x} \). Also find a \( n \) for which the difference in the solutions to \( x = T_n(x) \) and \( x = e^{-x} \) is less than 0.001. Assume that one real solution to \( x = T_n(x) \) remain in \([0.5, 0.61] \) for all \( n \geq 2 \).

Example 21. Find the global maximums of
1. \( w(x) \) on \([2, 6] \). \( w(x) = (x - 2)(x - 3)(x - 4)(x - 6) \)
2. \( f''(x) \) on \([0, 1] \). \( f(x) = \sin(x^2) \)
3. \( f^{(4)}(x) \) on \([0, 1] \). \( f(x) = \sin(x^2) \)

Casio 4. Fixed Point Method
ALPHA X ALPHA = \( e^\Box \) -ALPHA X CALC =

Mathematica 5. Fixed Point Method
Algorithm
\( g[x_] := E(-x); x = 0; For[k = 0, k <= 19, k++, \{x = g[x], Print[N\{k, x, Abs[x - g[x]]\}], 10]]} \)

Builtin function
FindRoot[\( f[x] == 0 \), \{x, 0\}], by iterations starting \( x_0 = 0 \)
Note 6. See the note AllRoots.pdf on finding the complex roots of \( x^5 - x - 1 = 0 \) using the Fixed Point method.

Definition 9. Newton’s method for finding roots of \( f(x) = 0 \)
\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

Note 7. We can analyze the Newtons method as a Fixed Point Method with \( g(x) = x - \frac{f(x)}{f'(x)} \) when \( f'(x) \neq 0 \). Then \( g'(x) = \frac{f(x)f''(x)}{(f'(x))^2} \) indicates that local convergence is guaranteed.

Theorem 35. Local convergence of the Newton’s Method
Let \( z \) be a root of \( f \). If \( f \in \mathbb{C}^2 \) and \( f'(z) \neq 0 \) then there exists a neighbourhood of \( z \) where the Newton’s method is converging.

Theorem 36. Global convergence of the Newton’s method (Newton-Kantorovich Theorem, see Proofs.pdf for the proof)
1. \( f : [a, b] \rightarrow \mathbb{R} \)
2. \( f' \neq 0 \) and there exists \( \beta > 0 \) such that \( \frac{1}{|f'(x)|} \leq \beta \)
3. \( f' \) is Lipschitz continuous with constant \( \gamma \)
4. \( x_0 \in [a, b] \) and \( x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0 \)
5. \( \left| \frac{f(x_0)}{f'(x_0)} \right| = \alpha \)
6. \( q = \alpha \beta \gamma < \frac{1}{2} \)
7. \( [x_0 - 2\alpha, x_0 + 2\alpha] \subset [a, b] \)
Then
1. \( \lim_{k \to \infty} x_k = z \in [x_0 - 2\alpha, x_0 + 2\alpha] \) is a unique root of \( f \)
2. \( |x_k - z| \leq 2\alpha q^{k-1} \)

Example 22. Consider the Newton’s method of finding the real roots of \( x - e^{-x} \) and \( x^5 - x - 1 = 0 \)
1. Treat the method as an fixed point method and find the no of iterations needed to calculate the root to an accuracy of \( 10^{-4} \) and find the root.
2. Use the error formula above for the Newton’s method and find the no of iterations needed to calculate the root to an accuracy of \( 10^{-4} \) and find the root.
3. Use more terms in the Taylor series (instead of 2 terms used in the Newton’s method) and propose a possibly faster method to find the root.
4. If \( f \) was not differentiable, propose a method which uses the secant (instead of the tangent) joining two successive points.
5. Try to find complex roots using the Newton’s method (see Note 3).

Example 23.
1. Do Example 5 for \( \sin x = 2x \) and \( x = e^{-x} \)
2. Try to solve \( x^m = 0 \) for \( m \in \mathbb{R} \). What is going wrong/right?.
3. Show that the sequence \( x_{k+1} = \frac{x_k}{2} + \frac{a}{2x_k} \) converges to \( \sqrt{a} \), provided we select \( x_0 \) on a suitable range. What is such a range?
4. Suppose we want to solve \( \tan^{-1}x = 0 \) by the Newton’s method. Find the value \( z \) such that the Newton’s method is converging for \( 0 < x_0 < z \), diverging for \( x_0 > z \).
and enters into a cycle for $x_0 = z$.

5. Your CASIO calculator can integrate, $\int_a^b f(x)\,dx$ is evaluated as $\int (f(x), a, b)$. Find the $z$ value for which $P(x < z) = 0.8$ when $X \sim N(0,1)$, i.e. when $X$ is Normally distributed with mean 0 and standard deviation 1 which is having a PDF $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

6. Find the height of the circular sector with arc length $2x$ and chord length $x$.

7. What is the height if the shape is a parabola?

**Example 24.** An iterative method of finding solutions to a non-linear equation $f(x) = 0$ is said to have a convergence of order $p$ iff $|x_{k+1} - z| \leq r|x_k - z|^p$ where $x_k$ is the $k$th iteration, $z$ is the solution and $r$ is a constant. Show that $p = 1$ for the fixed point method and $p = 2$ for the Newton’s method.

**Casio 5.** Solving cubic $x^3 + 2x^2 + 3x + 4 = 0$ with roots $x_1, x_2, x_3$

**MODE 5: EQN 4:**

\[ ax^3 + bx^2 + cx + d \]

1 = 2 = 3 = 4 == $x_1 = x_2 = x_3$

**Mathematica 6.**

\[
\text{NRoots}[x^5 - x - 1 == 0, x]
\]

\[
\text{NSolve}[f[x] == 0, x]
\]

\[
\text{FindRoot}[f[x] == 0, \{x, x0}\]
\]
8 Bivariable Real Analysis

Definition 10. Functions of two variables \( f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \)

Example 25. Draw the graphs of the following functions
1. \( f(x, y) = x^2 + y^2 \)
2. \( f(x, y) = \sqrt{x^2 + y^2} \)
3. \( \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1 \)

Definition 11. Limit
\[
\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L \iff \\
\forall \epsilon > 0 \exists \delta > 0 \forall (x, y), 0 < d((x, y), (a, b)) < \delta \Rightarrow |f(x, y) - L| < \epsilon
\]

Note 8. Matric
\( d : \mathbb{R}^2 \rightarrow [0, \infty) \) is a distance measuring function, called a Matric in \( \mathbb{R}^2 \). Some options are
1. \( \sqrt{(x-a)^2 + (y-b)^2} \)
2. \( |x-a| + |y-b| \)
3. \( \max\{|x-a|, |y-b|\} \)

One can show all these are matrices. We will use the first matrix.

0 < \( d((x, y), (a, b)) < \delta \) is an open set (without its boundary) around and excluding (a, b). Such a set is called a deleted neighborhood which we symbolize by DNbd.

Example 26. Use the definition to show that \( \lim_{(x,y) \rightarrow (2,3)} xy = 6 \)

Theorem 37. If \( \lim_{(x,y) \rightarrow (a,b)} f(x, y) = L, \lim_{x \rightarrow a} f(x, y) = g(y) \) in a DNbd of b and \( \lim_{y \rightarrow b} f(x, y) = h(x) \) in a DNbd of a, Then \( \lim_{x \rightarrow a} h(x) = \lim_{y \rightarrow b} g(y) = L \).

Example 27. Prove by definition that if \( \lim_{(x,y) \rightarrow (0,0)} f(x, y) \) along \( y = x \) and \( y = 2x \) are different, then the limit is not existing.

Example 28. Investigate the existence of the limit, \( \lim_{(x,y) \rightarrow (0,0)} \) for the following functions.

1. \( f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) = (0, 0) \\ 0, & (x, y) \neq (0, 0) \end{cases} \)
2. \( f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \)
3. \( f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \)
4. \( f(x, y) = \begin{cases} x \sin \frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases} \)

Definition 12. Continuity of \( f \) at \( (a, b) \) i.e. \( f \in C(a, b) \)

\( \lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b) \)

Definition 13. Partial Derivatives
\[
\frac{\partial f}{\partial x}(a, b) = \lim_{x \rightarrow a} \frac{f(x, y) - f(a, b)}{x-a} = \lim_{x \rightarrow a} \frac{f(a+\Delta x, y) - f(a, y)}{\Delta x} = f_x(a, b)
\]
\[
\frac{\partial f}{\partial y}(a, b) = \lim_{y \rightarrow b} \frac{f(x, y) - f(a, b)}{y-b} = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{\Delta y} = f_y(a, b)
\]

Definition 14. \( f \in C^1 \iff f_x \in C \) and \( f_y \in C \)

Theorem 38. Mean Value Theorem
1. \( f : \mathbb{D} \rightarrow \mathbb{R}, \mathbb{D} = \{(x, y)|(x-a)^2 + (y-b)^2 < \delta^2\} \)
2. \( f_x \) and \( f_y \) exists on \( \mathbb{D} \)
3. \( \Delta x^2 + \Delta y^2 < \delta^2 \)

Then
1. \( f(a + \Delta x, b + \Delta y) = \Delta xf_x(a + \theta \Delta x, b) + \Delta yf_y(a + \Delta x, b + \alpha \Delta y) \)
2. \( 0 < \theta, \alpha < 1 \)
Definition 15. Differentiability of \( f \) at \( (a, b) \) i.e. \( f \in \mathbb{D}(a, b) \)
1. \( f_x, f_y \) exists at \( (a, b) \)
2. \( f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b) \Delta x + f_y(a, b) \Delta y + o(\Delta x, \Delta y) \) for all \( \Delta x^2 + \Delta y^2 \ll \) and for some \( \phi, \psi \)
3. \( \lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{\phi(\Delta x, \Delta y)}{\Delta x, \Delta y} = \lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{\psi(\Delta x, \Delta y)}{\Delta x, \Delta y} = 0 \)

Theorem 39. \( f \in C^1 \Rightarrow f \in \mathbb{D} \Rightarrow f \in C \)

Example 29. Let \( f(x, y) = g(\sqrt{x^2 + y^2}), g(x) = x \sin \frac{1}{x}, g(0) = 0 \). Show that \( f \in \mathbb{D} \) but \( f \notin C \)

Theorem 40. Chain Rule. \( f = f(x, y) \in C^1 \).
1. If \( y = y(t), x = x(t) \in C^1 \) then \( \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \)
2. \( y = y(u, v), x = x(u, v) \in C^1 \) then \( \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \) and \( \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \)

Note 9. The above may be written as
\[
\frac{df}{dt} = \left( \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{array} \right) \left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \frac{\partial f}{\partial (x, y)} \left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) \quad \text{and} \quad \frac{\partial f}{\partial (u, v)} = \left( \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right) \left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \frac{\partial f}{\partial (x, y)} \frac{\partial (x, y)}{\partial (u, v)}
\]

With \( x = \left( \begin{array}{c} x \\ y \end{array} \right) \) and \( u = \left( \begin{array}{c} u \\ v \end{array} \right) \), the above may also be written as
\[
(f \circ x)'(t) = (f' \circ x)(t)x'(t) \quad \text{and} \quad (f \circ x)'(u) = (f' \circ x)(u)x'(u)
\]

We also see that \( \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial (x, y)} = f'(x) \) is acting as the true derivative of \( f = f(x, y) \). Therefore it is called the Gradient of \( f \) or \( \nabla f = \text{grad} f \)

The determinant, \( \text{det} \frac{\partial (x, y)}{\partial (u, v)} = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \) is called the Jacobian

Definition 16. Directional Derivative of \( f \) in the direction of the non-zero unit vector \( u = (u, v) \) at \( a = (a, b) \) is \( D_u f(a, b) = \lim_{\Delta t \to 0} \frac{f(a + u \Delta t, b + v \Delta t) - f(a, b)}{\Delta t} \)

Theorem 41. \( f \in C^1, \nabla f(a, b) \neq 0 \)
1. \( D_u f(a, b) = \frac{\partial f}{\partial x}(a, b) u + \frac{\partial f}{\partial y}(a, b) v = \nabla f(a, b) u \)
2. \( \max_u D_u f(a, b) = D_{\nabla f(a, b)} f(a, b) = ||\nabla f(a, b)|| \)
3. \( \min_u D_u f(a, b) = D_{-\nabla f(a, b)} f(a, b) = -||\nabla f(a, b)|| \)

Theorem 42. For the surface \( f = f(x, y) \in C^1 \) at \( (a, b) \)
1. Normal vector : \( \mathbf{n}(a, b) = (f_x(a, b), f_y(a, b), -1) = (\nabla f(a, b), -1) \)
2. Equation of the Tangent Plane:
\[
z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) = \nabla f(a, b) \left( \begin{array}{c} x-a \\ y-b \end{array} \right) = f(a) + \nabla f(a)(x-a)
\]

Example 30. Let \( f(x, y) = x^4 + y^4 - x^2 - y^2 + 1 \). At \( (1, 2) \) find
1. Direction at which the function is increasing most rapidly
2. Directional derivative in that direction
3. Equation of the tangent plane

Example 31. Assume that all functions are \( C^1 \)
1. Show that if \( x = x(u, v), y = y(u, v), u = u(r, s), v = v(r, s) \) then
\[
\frac{\partial (x, y)}{\partial (r, s)} = \frac{\partial (x, y)}{\partial (u, v)} \frac{\partial (u, v)}{\partial (r, s)}
\]
2. Show that if \( u = f(x, y), v = g(x, y) \) then a functional relation of the form \( h(u, v) = 0 \) exists iff the Jacobian is identically zero.
Definition 17. Higher Order Derivatives
\[ f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \]
\[ f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \]
\[ f_{yx} = (f_y)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \]
\[ f_{yy} = (f_y)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \]

Note 10.
1. We write \( f \in \mathcal{C}^2 \) to mean that \( f_{xx}, f_{xy}, f_{yx}, f_{yy} \in \mathcal{C} \)
2. In a similar manner we write \( f \in \mathcal{C}^n \) to mean that all the \( n \) th order partial derivatives are continuous. There are \( 2^n \) of them.
3. There are \( ^n C_m = \frac{n!}{m!(n-m)!} \), \( n \) th order partial derivatives that contains \( x,m \) times.

Example 32. Let \( f(x, y) = \begin{cases} xy^2 - y^2 & , (x, y) = (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases} \)
Show that \( f_{xy}(0, 0) \neq f_{yx}(0, 0) \)

Theorem 43. \( f \in \mathcal{C}^2 \Rightarrow f_{xy} = f_{yx} \)

Example 33. If \( u = u(x, y) \in \mathcal{C}^2 \) then prove that the Laplace operator \( \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) can be written as \( \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \)

Theorem 44. Taylor series of \( f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^{n+1} \) at \( (a, b) \).
\[ f(a + h, b + k) = \sum_{m=0}^{n} \frac{1}{m!} D^m f(a, b) + \frac{1}{(n+1)!} D^{n+1} f(a + \theta h, b + \theta k) \]
where \( D = h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}, D^m = (D^m)^{-1}, D^0 = 1 \) and \( \theta \in (a, b) \)

Note 11. When \( n = 2 \) we may write the above as
\[ f(a + h) = f(a) + \nabla f(a) \cdot h + \frac{1}{2} h^T H f(c) h \]
where \( \nabla f = (f_x, f_y) \) is the Gradient and \( H f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \) is the Hessian.

and \( a = \begin{pmatrix} a \\ b \end{pmatrix} \), \( h = \begin{pmatrix} h \\ k \end{pmatrix} \), \( c = a + \theta h, \theta \in (0, 1) \).

Note that the first two terms are the Tangent Plane at \( a \).
Also \( h^T H f(c) h = h^2 f_{xx}(c) + 2hk f_{xy}(c) + k^2 f_{yy}(c) = h^2 f_{xx}(c) \left( \left( \frac{k}{h} \right)^2 + \frac{\det H f(c)}{(f_{xx}(c))^2} \right) \)
if \( f_{xx}(c) \neq 0 \). Note that \( \det H f = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - (f_{xy})^2 \) when \( f \in \mathcal{C}^2 \).
Also \( \text{tr} H f = f_{xx} + f_{yy} \) is the Trace of the matrix \( H f \).

Then
\[ f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^2. \nabla f(a) = 0. \]
\[ \text{tr} H f(a) > 0 \text{ and } \det H f(a) > 0 \]
\[ \iff f_{xx}(a) > 0 \text{ and } \det H f(a) > 0 \]
\[ \Rightarrow f_{xx}(c) > 0 \text{ and } \det H f(c) > 0, \text{ for sufficiently small } h \]
\[ \Rightarrow f(a + h) - f(a) = \frac{1}{2} h^T H f(c) h, \text{ for sufficiently small } h \]

Definition 18.
1. \( f \) has a relative minimum at \( a \) iff \( f(a) > f(a + h) \) in a nbd of \( a \).
2. \( f \) has a relative maximum at \( a \) iff \( f(a) < f(a + h) \) in a nbd of \( a \).
3. \( f \) has a saddle point at \( a \) iff both \( f(a) > f(a + h) \) and \( f(a) < f(a + h) \) in a nbd of \( a \).
Theorem 45. \( f \in \mathcal{C}^2 \) and \( \nabla f(a) = 0 \)
1. If \( \det Hf(a) > 0 \) and \( \text{tr} Hf(a) > 0 \) then \( a \) is a local minimum of \( f \).
2. If \( \det Hf(a) > 0 \) and \( \text{tr} Hf(a) < 0 \) then \( a \) is a local maximum of \( f \).
3. If \( \det Hf(a) < 0 \) then \( a \) is a saddle point of \( f \).

Definition 19. \( a \) is a Critical Point of \( f \) iff \( \nabla f(a) = 0 \) of undefined

Example 34. Find the critical points and determine the nature(max/min/saddle) of them
1. \( f(x, y) = x^3 - 12x + y^3 - 27y + 5 \)
2. \( f(x, y) = x^4 + y^4 - x^2 - y^2 + 1 \)
3. \( f(x, y) = x^4 + y^4 \)
4. Propose a method if \( \det Hf = 0 \) when \( \nabla f = 0 \)

Theorem 46. Constrained Optimization/Lagrange Multipliers
Let \( f, g \in \mathcal{C}^1 \) and \( \nabla g \neq 0 \). Then the maxima/minima of \( f(x, y) \) subjected to \( g(x, y) = 0 \) are included in each of
1. \( \det \frac{\partial(f,g)}{\partial(x,y)} = 0 \) and \( g(x, y) = 0 \)
2. \( \nabla f(x, y) = \lambda \nabla g(x, y) \) and \( g(x, y) = 0 \)

Example 35.
1. Find the shortest distance from the point \( 1,0 \) to the parabola \( y^2 = 4x \).
2. Substitute \( y^2 = 4x \) or \( x = y^2/4 \) and minimize the distance function in 1. as a function of \( y \) or \( x \). Explain why we get/can’t get the answer in 1.
3. Find the absolute maximum/minimum of \( x^4 + y^4 - x^2 - y^2 + 1 \) on the disk \((x - 1)^2 + y^2 \leq 4\)
4. Find the directions of the axes of the ellipse \( 5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0 \)

9 Least Square Polynomial

Theorem 47. Least Square Line \( y = ax + b \) for the points \( (x_k, y_k), 1 \leq k \leq n \)
That minimizes \( E(a,b) = \sum_{k=1}^{n}(ax_k + b - y_k)^2 \) is given by

\[
\begin{pmatrix}
\sum_{k=1}^{n} x_k^2 & \sum_{k=1}^{n} x_k \\
\sum_{k=1}^{n} x_k & \sum_{k=1}^{n} 1
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} =
\begin{pmatrix}
\sum_{k=1}^{n} x_k y_k \\
\sum_{k=1}^{n} y_k
\end{pmatrix}
\]

This can also be written as \( X^T X \begin{pmatrix} a \\ b \end{pmatrix} = X^T Y \) where
\( X^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \) and \( Y^T = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} \)

Theorem 48. Properties of the Least Square Line
With \( \bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k, \bar{y} = \frac{1}{n} \sum_{k=1}^{n} y_k, s_{xx} = \frac{1}{n} \sum_{k=1}^{n} (x_k - \bar{x})^2 = \frac{1}{n} \sum_{k=1}^{n} x_k^2 - \bar{x}^2, \) and
\( s_{xy} = \frac{1}{n} \sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \frac{1}{n} \sum_{k=1}^{n} x_k y_k - \bar{x} \bar{y} \)
We have \( a = \frac{s_{xy}}{s_{xx}} \)
Also \((\bar{x}, \bar{y})\) is on the Least square line and therefore \( \bar{y} = a \bar{x} + b \) or \( b = a \bar{x} - \bar{y} \)
If \( HE = \begin{pmatrix} E_{aa} & E_{ab} \\ E_{ba} & E_{bb} \end{pmatrix} \) is the Hessian of \( E(a,b) \) we have \( HE = 2X^T X \) and 
\[
\det HE = E_{aa}E_{bb} - (E_{ab})^2 = 4n^2s_{xx} > 0 \quad \text{when } x_k \text{ are different and}
\]
\[
\text{tr} HE = E_{aa} + E_{bb} = 2\sum x_k^2 + 2n > 0 \quad \text{so } (a,b) \text{ is a global minimum.}
\]

**Example 36.** Let \( A = \{(2,1), (3,2), (4,3), (6,4)\} \)
1. Find the Lest Square Line for \( A \).
2. Show that the least square parabola \( y = ax^2 + bx + c \) for the data set \((x_k, y_k), k = 1, 2, \ldots, n\) is given by
\[
\begin{pmatrix}
\sum x_k^4 \\
\sum x_k^3 \\
\sum x_k^2 \\
\sum 1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= 
\begin{pmatrix}
\sum x_k^2 y_k \\
\sum x_k y_k
\end{pmatrix}
\]
3. Find the Least Square Parabola for \( A \)

**Example 37.** Let \( A = \{(1,1), (2,1), (3,2), (4,3)\} \)
1. Find the least square polynomials of degree 0,1,2,3,4 for \( A \) if it is possible.
2. Calculate the exact error before and after finding the coefficients in each case.
3. Show that we have a unique solution when each \( x_k \) is different.
4. Fit a least square function of the form \( y = ax + bx^3 + cx^4 \) for \( A \).
5. Fit a least square function of the form \( y = ae^x + b\sin x + c\cos x \) for \( A \).
6. Find the best combination of functions out of \( \{1, x, x^2, e^x, \sin x, \cos x, \log x\} \) if we are looking for a combination of 3 functions.
7. Show that the Correlation Coefficient given by \( r = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}} \) is a measure of the linearity of data in the case of the least square line.
8. Suppose we have a 3D data set \( B = \{(1,1,1), (2,1,2), (3,2,3), (4,3,4)\} \). Propose a Lagrange-type two variable polynomial and a least square plane.

**Casio 6.** CASIO fx-991ES

MODE 3:STAT 3:+ cx^2 2 = 3 = 4 = 6 = REPLAY UP 1 = 2 = 3 = 4 = SHIFT

STAT(1) 1:Type, 2: Data, 3:Edit, 4:Sum(1:\( \sum x^2 \), 2:\( \sum x \), 4: \( \sum y \), 5: \( \sum xy \), 6: \( \sum x^3 \), 7: \( \sum x^2 y \), 8: \( \sum x^4 \)), 7:Reg(1:A, 2:B, 3:C)

**Mathematica 7.** Fit[{{2,1}, {3,2}, {4,3}, {6,4}}, \{x^2, x, 1\}, x]
10 Ordinary Differential Equations, ODEs

Definition 20. First Order ODE \( \frac{du}{dx} = f(x,y), y(x_0) = y_0 \)

Theorem 49. Cauchy-Peano (Existence of solutions)
If \( f \in C \) then the above ODE has a \( C^1 \) solution in a nbd of \((x_0, y_0)\)

Theorem 50. Picard-Lindelof (Uniqueness of solutions)
If \( f \in C \) and \( f \in LC \) in \( y \) uniformly in \( x \). Then the above ODE has a unique \( C^1 \) solution in a nbd of \((x_0, y_0)\)

Definition 21. \( f \in LC \) in \( y \) uniformly in \( x \)
iff \( \exists L > 0 \forall x \forall y_1, y_2; |f(x, y_1) - f(x, y_2)| < L |y_1 - y_2| \)

Theorem 51. If \( \frac{∂f}{∂y} \in C \) in a closed bounded set on \( \mathbb{R}^2 \) then \( f \in LC \) in \( y \) uniformly in \( x \).

Example 38. Discuss the Existence and Uniqueness of the following ODEs.
1. \( \frac{dy}{dx} = y^3, y(0) = 0 \)
2. \( \frac{dy}{dx} = y^{1/3}, y(0) = 0 \)

Definition 22. Variable Separable: \( f(x,y) = \frac{g(x)}{h(y)}; g, h \in C; h(y) \neq 0 \)
\( \int h(y)dy = \int g(x)dx \)

Definition 23. \( \frac{dy}{dx} = \frac{x}{y}, y(0) = \sqrt{2} \)

Definition 24. Homogeneous:
\( y = vx, v = v(x), f(x,vx) = g(v) \in C, g(v) \neq v, x \neq 0 \)
\( \frac{dy}{dx} = v + x \frac{dx}{dx} = g(v) \Rightarrow \frac{dx}{dx} = \frac{g(v)-v}{x} \): Variable Separable

Example 39. \( \frac{dy}{dx} = \frac{x^2+y^2}{xy}, y(1) = 1 \)

Definition 25. Linear: \( f(x,y) = Q(x) - P(x)y; P, Q \in C \)
Integrating Factor: \( I(x) = e^{\int P(x)dx} \Rightarrow \frac{dx}{dx}(I(x)y) = I(x)Q(x) \Rightarrow y = \frac{1}{I(x)} \int Q(x)I(x)dx \)

Example 40. \( \frac{dy}{dx} + \frac{y}{x} = \log x, y(1) = 1 \)

Definition 26. Bournoulli: \( f(x,y) = Q(x)y^n - P(x)y; P, Q \in C \)
\( z = y^{1-n} \Rightarrow \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \): Linear

Example 41. \( \frac{dy}{dx} + \frac{y}{x} = y^3 \log x, y(1) = 1 \)

Definition 27. Exact:
\( M(x,y) + N(x,y) \frac{dy}{dx} = 0; N(x,y) \neq 0; M, N \in C^1; \frac{∂M}{∂y} = \frac{∂N}{∂x} \in C \)
Then there exists \( f \) such that \( \frac{df}{dx} = M \) and \( \frac{df}{dy} = N \) (see proof)
\( \Rightarrow 0 = M + N \frac{dy}{dx} = \frac{df}{dx} + \frac{∂f}{∂y} \frac{dy}{dx} = \frac{df}{dx} \Rightarrow f = c \): constant

Proof. Let \( \frac{∂M}{∂y} \) = \( \frac{∂N}{∂x} \) = \( P(x,y) \in C \).
Then \( M(x,y) = \int P(x,y)dy \) and \( N(x,y) = \int P(x,y)dy \)
Now \( \int M(x,y)dx = \int \int P(x,y)dydx \)
\[ \int \int P(x,y) dx dy, \text{ by Fubini's Theorem for Double Integrals since } P \in \mathcal{C} \]
\[ = \int N(x,y) dy = f(x,y), \text{ say Then } \frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N. \]
Also note that \( \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = P \) and \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} = P. \)
So \( \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \) automatically. \( \Box \)

**Example 42.**
1. \((y^2 + x^2) + (2xy + y) \frac{dy}{dx}, y(1) = 1\)
2. \((3x^2 + 6xy^2) + (6x^2y + 4y^3) \frac{dy}{dx} = 0\)

**Definition 28.** Reducible to Exact:
\[ M(x,y) + N(x,y) \frac{dy}{dx} = 0, M, N \in \mathcal{C}^1, \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \in \mathcal{C} \]
If \( \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)/N = g(x) \) is a function of \( x \) alone,
define \( I(x) = e^{\int g(x) dx} \) so \( \frac{\partial (NI)}{\partial x} = I \frac{\partial N}{\partial y} + NIg(x) = I \frac{\partial M}{\partial y} = \frac{\partial (MI)}{\partial y} \): Exact
If \( \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)/M = h(y) \) is a function of \( y \) alone,
define \( J(y) = e^{\int h(y) dy} \) so \( \frac{\partial (MJ)}{\partial x} = J \frac{\partial M}{\partial y} + MJh(y) = J \frac{\partial N}{\partial x} = \frac{\partial (NJ)}{\partial x} \): Exact

**Example 43.**
1. \((x^3 + y^3) - xy^2 \frac{dy}{dx} = 0\)
2. \(y - (2x + y) \frac{dy}{dx}\)

**Definition 29.** Second Order ODE: \( \frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}), y(x_0) = y_0, y'(x_0) = y'_0\)

**Definition 30.** Second Order Linear ODE:
\[ \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x), y(x_0) = y_0, y'(x_0) = z_0 \]
\[ \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0 \text{ is called the Homogeneous Equation and} \]
\[ \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x) \text{ is called the Non-Homogeneous Equation} \]

**Theorem 52.**
1. If \( p, q, r \in \mathcal{C} \) then the above ODE has a unique solution in a nbd of \((x_0, y_0)\)
2. The solution can be expressed as \( y = y_c + y_p \)
3. The Complimentary/Fundamental Solution \( y_c \) is the solution for the corresponding Homogeneous Equation.
4. It can be expressed as \( y_c = au(x) + bv(x) \) where \( a, b \) are constants and \( u, v \) are the Fundamental Solutions which are Linearly Independent, i.e.
\( \forall (a, b) [\forall x (au(x) + bv(x) = 0) \Rightarrow (a, b) = (0, 0)] \)
5. Note that \( u, v \) satisfy the Homogeneous equation on their own so is their Linear Combination \( au(x) + bv(x) \)
6. Particular Solution \( y_p = w(x) \) is the solution for the corresponding Non-Homogeneous Equation. It does not have an arbitrary constants as in the case of \( y_c \).

**Definition 31.** Second Order Linear ODE with constant coefficients:
\[ \frac{d^2 y}{dx^2} + p \frac{dy}{dx} + qy = r(x); p, q \text{ are real numbers} \]

**Theorem 53.**
Method 1
By substituting \( y = ce^{\alpha x} \) in the Homogeneous Equation we arrive at the Characteristic Equation \( \alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = 0 \). There are three possibilities
1. \( a, b \in \mathbb{R}, a \neq b \Rightarrow y_c = ce^{ax} + de^{bx}, u(x) = e^{ax}, v(x) = e^{bx} \)
2. \( a, b \in \mathbb{R}, a = b \Rightarrow u(x) = e^{ax} \). To get the other solution we let \( y = xe^{ax} \) and confirm that it satisfies the Homogeneous Equation. So \( v(x) = xe^{ax} \) and \( y_c = ce^{ax} + dxe^{ax} \)
3. \( a, b \in \mathbb{C}, a = a_1 + ia_2 = b \Rightarrow y_c = ce^{a_1x} \sin a_2x + de^{a_1x} \cos a_2x, u(x) = e^{a_1x} \sin a_2x, v(x) = e^{a_1x} \cos a_2x \)

The Particular Solution \( y_p \) may be obtained by assuming a solution and confirming it.

**Method 2**

If \( \alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = \alpha(\alpha - b) - a(\alpha - b) = \alpha(\alpha - a) - b(\alpha - a) \)

We can write \( \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \frac{dy}{dx}(\frac{dy}{dx} - ay) - b(\frac{dy}{dx} - ay) = \frac{dz}{dx} - bz = r(x) \) and \( \frac{dy}{dx} - ay = z \) or alternatively
\[
\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \frac{d}{dx}(\frac{dy}{dx} - by) - a(\frac{dy}{dx} - by) = \frac{dz}{dx} - az = r(x) \text{ and } \frac{dy}{dx} - by = z
\]

In either case we get two First Order Linear ODEs which can be solved to find \( z \) and then \( y \).

**Example 44.** Solve
1. \( \frac{dv}{dx} - 5\frac{dy}{dx} + 6y = \sin x \)
2. \( \frac{dv}{dx} - 2\frac{dy}{dx} + y = e^x \)

**Definition 32.** Reducible to Second Order Linear ODE with constant coefficients:
\( x^2 \frac{d^2y}{dx^2} + px \frac{dy}{dx} + qy = r(x); p, q \) are real numbers

Let \( x = e^z \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d}{dz}(\frac{dy}{dz}) = \frac{dy}{dz} + p\frac{dy}{dx} + qy = r(x) \Rightarrow \frac{d^2y}{dz^2} + (p - 1)\frac{dy}{dx} + qy = r(e^z) \): Second Order Linear ODE with constant coefficients

**Definition 33.** Wronskian
\[
W(u, v)(x) = uv' - vu' = \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}
\]

**Theorem 54.** If \( u, v \) are Fundamental Solutions to the Homogeneous Equation, then
1. \( W' + p(x)W = 0 \)
2. \( W(u, v)(x) = ce^{-\int p(x)dx} = W(u, v)(x_0)e^{-\int_{x_0}^{x} p(t)dt} \)
3. \( \forall x, W(u, v) = 0 \Longleftrightarrow u, v \) are Linearly Independent.

**Example 45.**
1. Assume \( v(x) = c(x)u(x) \) and derive a method of finding \( c(x) \). Are the results same as above?
   1. We know that \( u(x) = e^{ax} \) is a solution to \( \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0 \). Show that \( v(x) = xe^{ax} \) is the other Linearly Independent Solution.
   2. Consider the ODE: \( x^2y'' + x(x+1)y' - y = 0 \). If \( u(x) = \frac{e^{-x}}{x} \) is a solution, use Wronskian to find the other linearly independent solution \( v(x) \).

**Theorem 55.** If \( u, v \) are fundamental solutions to the Homogeneous Equation then
1. Particular Solution can be expressed as \( y_p = c(x)u(x) + d(x)v(x) \)
2. \[
\begin{pmatrix}
  u & v \\
  u' & v'
\end{pmatrix}
\begin{pmatrix}
  c' \\
  d'
\end{pmatrix} = 
\begin{pmatrix}
  0 \\
  r
\end{pmatrix}
\]

3. \[c' = -rv \quad \text{and} \quad d' = ru \quad \text{and} \quad W = \frac{1}{r} \det \begin{pmatrix}
  u & 0 \\
  u' & r
\end{pmatrix} \]

4. \[y_p(x) = \int_{x_0}^{x} \frac{v(x)u(t)-u(x)v(t)}{W(u,v)(t)} dt\]

**Example 46.**

1. Show that \[u(x) = x\] is a solution to \[y'' - xy' + y = 0\].
2. Find the other linearly independent solution \[v(x)\] and express it in terms of well-known function and the error function given by \[\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt\].
3. Solve \[y'' - xy' + y = 1\] and express the solution by well-known function and the error function.
4. Solve \[y'' - xy' + y = 1, y(1) = 2\].

**Example 47.**

Each of the following are well-known differential equations with a parameter \[n\] which is a positive integer. Find the intervals of \[x\] within which the solution exist. Select one \[n \geq 2\] and show that the \[u(x)\] given below is actually a solution. Also find the other linearly independent solution \[v(x)\].

1. Legendre ODE: \[(1-x^2)y'' - 2xy' + n(n+1)y = 0\];
   Legendre Polynomials: \[u(x) = P_n(x) = \frac{d^n}{dx^n}[(x^2-1)^n]\]
2. Laguerre ODE: \[xy'' + (1-x)y' + ny = 0\];
   Laguerre Polynomials: \[u(x) = L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} [e^{-x}x^n]\]
   Also try to find solutions directly by assuming power series of the form \[u(x) = \sum_{k=0}^{\infty} a_k x^k\] and finding \[a_k\]. What is the radius of convergence?

**Example 48.** Consider the ODE: \[y' + y = x, y(0) = 0\].

1. Find the solution analytically and \[y(1)\] in decimal.
2. Consider a Numerical Solution by the Euler’s Method. Let \[h\] be the width of a subdivision of \([0,1], x_k = kh\) and \[y_k\] is an approximation to \[y(x_k)\]. Consider the 1st order Taylor Series expansion of \[y(x_k + h)\] at \[x_k\] and show that the formula \[y(k+1) = y_k + h(kh - y_k)\] can be used for generating \[y_k\]. Use this formula to approximate the value of \[y(1)\] when \[h = 0.1\].
3. With the same setting as above consider the 2nd order Taylor Series expansion of \[y(x_k + h)\] at \[x_k\] and show that the formula \[y(k+1) = y_k + h(kh - y_k) + \frac{h^2}{2}(1 - (kh - y_k))\] can be used for generating \[y_k\]. Use this formula to approximate \[y(1)\] when \[h = 0.1\].

**Mathematica 8.**

\[
\text{DSolve}\{y''[x] + p[x]y'[x] + q[x]y[x] == r[x], y[a] == b, y'[a] == c\}, x, y[x]\]